A CLASSIFICATION OF PRIMITIVE PERMUTATION GROUPS WITH FINITE STABILIZERS

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Abstract. We classify all infinite primitive permutation groups possessing a finite point stabilizer, thus extending the seminal O'Nan–Scott Theorem to all primitive permutation groups with finite point stabilizers.

1. Introduction

Recall that a transitive permutation group G on a set Ω is primitive if the only G-invariant partitions are $\{\{\alpha\} : \alpha \in \Omega\}$ and $\{\Omega\}$. In the finite case, these groups are the fundamental actions from which all permutation groups are constituted.

The finite primitive permutation groups were classified by the famous O'Nan–Scott Theorem, proved independently by O'Nan and Scott [15], which describes in detail the structure of finite primitive permutation groups in terms of finite simple groups. A modern statement of the theorem with a self-contained proof can be found in [6].

Primitive permutation groups with finite point stabilizers are precisely those primitive groups whose subdegrees are bounded above by a finite cardinal (see [16]). This class of groups also includes all infinite primitive permutation groups that act regularly on some finite suborbit (see [9, Problem 7.51]). Because of this, it has become necessary to determine their precise structure. Our main result is Theorem 1.1 below; in conjunction with the O'Nan–Scott Theorem, this yields a satisfying classification of all primitive permutation groups with finite point stabilizers, describing in detail their structure in terms of finitely generated simple groups.

Theorem 1.1. *If $G \leq \mathrm{Sym}(\Omega)$ is an infinite primitive group with a finite point stabilizer G_α, then G is finitely generated by elements of finite order and possesses a unique (non-trivial) minimal normal subgroup M; there exists an infinite, non-abelian, finitely generated simple group K such that $M = K_1 \times \cdots \times K_m$, where $m \geq 1$ is finite and $K_i \cong K$ for $1 \leq i \leq m$; and G falls into precisely one of the following categories:*

(i) *M acts regularly on Ω, and G is equal to the split extension $M.G_\alpha$ for some $\alpha \in \Omega$, with no non-identity element of G_α inducing an inner automorphism of M;*

(ii) *M is simple, and acts non-regularly on Ω, with M of finite index in G and $M \leq G \leq \mathrm{Aut}(M)$;*

(iii) *M is non-regular and non-simple. In this case $m > 1$, and G is permutation isomorphic to a subgroup of the wreath product $H \,\mathrm{Wr}_\Delta\, \mathrm{Sym}(\Delta)$ acting in the product action on Γ^m, where $\Delta = \{1, \ldots, m\}$, Γ is some infinite set, $H \leq \mathrm{Sym}(\Gamma)$ is an infinite primitive group with a finite point stabilizer and H is of type (ii). Here K is the unique minimal normal subgroup of H.*

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For each type (i), (ii) and (iii) there exist examples of infinite primitive permutation groups with finite point stabilizers. We present these in Section 4.

This paper is not the first to extend the O'Nan–Scott Theorem to specific classes of infinite groups. In [7], a version of the O'Nan–Scott Theorem for definably primitive permutation groups of finite Morley rank is proved, and [4] contains O'Nan–Scott-style classifications for several classes of countably infinite primitive permutation groups in geometric settings. Macpherson and Praeger ([8]) extend the O'Nan–Scott Theorem to infinite primitive permutation groups acting on countably infinite sets that posses a minimal closed (in the topology of pointwise convergence) normal subgroup which itself has a minimal closed normal subgroup. Much of our proof of Theorem 1.1 involves establishing that an infinite primitive permutation group with a finite point stabilizer has a unique minimal normal subgroup which itself has a minimal normal subgroup. Thenceforth, we are at liberty to use the arguments contained in [8], and in particular the proof of [8, Lemma 4.7]. In doing so we shorten our argument establishing case (iii) of Theorem 1.1 considerably.

Of course it must also be mentioned that much of the strength of original O'Nan–Scott Theorem derives from the classification of the finite simple groups; nothing similar exists for finitely generated simple groups.

2. The socle

We use $\mathrm{Sym}(\Omega)$ to denote the full symmetric group of a set Ω and will write K^g to mean $g^{-1}Kg$ throughout. In fact, we will always write group actions in this way, with α^g representing the image of $\alpha \in \Omega$ under $g \in G$ whenever G acts on Ω. The orbits of a normal subgroup of a group $G \leq \mathrm{Sym}(\Omega)$ are G-invariant partitions of Ω; thus, every non-trivial normal subgroup of a primitive permutation group G acts transitively on Ω. A transitive group G is primitive if and only if for all $\alpha \in \Omega$ the point stabilizer G_α is a maximal subgroup of G. Note that if G is transitive and G_α is finite for some $\alpha \in \Omega$, then G_α is finite for all $\alpha \in \Omega$. We say a group G is *almost simple* if there exists a normal non-abelian simple subgroup N such that $N \leq G \leq \mathrm{Aut}(N)$.

A *minimal normal subgroup* of a non-trivial group G is a non-trivial normal subgroup of G that does not properly contain any other non-trivial normal subgroup of G. The *socle* of G, denoted by $\mathrm{soc}(G)$, is the subgroup generated by the set of all minimal normal subgroups of G. If G has no minimal normal subgroup then $\mathrm{soc}(G)$ is taken to be $\langle 1 \rangle$. Non-trivial finite permutation groups always have a minimal normal subgroup, so their socle is non-trivial; this is not true in general of infinite permutation groups.

In this section we show that every infinite primitive permutation group G with finite point stabilizers has a unique minimal normal subgroup M, and M is characteristically simple, finitely generated, and of finite index in G. Furthermore, M is equal to the direct product of finitely many of its simple subgroups. Since M is the only minimal normal subgroup of G, it is necessarily equal to the socle of G.

Lemma 2.1. *If $G \leq \mathrm{Sym}(\Omega)$ is primitive, then $|G : N| \leq |G_\alpha|$ for all non-trivial normal subgroups $N \lhd G$ and all $\alpha \in \Omega$.*

Proof. Let $G \leq \mathrm{Sym}(\Omega)$ be primitive. If $N \lhd G$ is non-trivial, then it is transitive, and so for all $\alpha \in \Omega$ we have $G = NG_\alpha$. But this implies that G_α contains a right transversal of N in G, and therefore $|G : N| \leq |G_\alpha|$. □

If G is a primitive group of permutations of a set Ω, and some point stabilizer G_α is finite, then all point stabilizers are finite. Hence the suborbits of G are

all finite and it is known (see for example [1, Remark 29.8]) that Ω is countable. Furthermore, if some point stabilizer G_α of G is finite, it is simple to see that G is finitely generated by elements of finite order: the group G_α is a maximal subgroup of G, and for $\beta \in \Omega \setminus \{\alpha\}$ we have $\langle G_\alpha, G_\beta \rangle = G$ or G_α. Because G is primitive, the latter occurs only if G is regular, but infinite primitive permutation groups are never regular.

Theorem 2.2. *If $G \leq \mathrm{Sym}\,(\Omega)$ is primitive and infinite, and G_α is finite for some $\alpha \in \Omega$, then G has a unique (non-trivial) minimal normal subgroup M, and*

(i) *$|G : M| \leq |G_\alpha|$; and*
(ii) *M is finitely generated and characteristically simple.*

Proof. By Lemma 2.1, all non-trivial normal subgroups of G are of finite index in G. And because G is finitely generated, Hall's Theorem ([5, Theorem 4]) guarantees that there are only finitely many non-trivial normal subgroups of G. Thus, if M is the intersection of all non-trivial normal subgroups of G, then M has finite index in G by Poincaré's Theorem (see [13, Theorem 1.3.12] for example). The group M is therefore non-trivial and is thus the unique minimal normal subgroup of G.

By Lemma 2.1, the index of all non-trivial normal subgroups of G is bounded above by $|G_\alpha|$, so $|G : M| \leq |G_\alpha|$. Schreier's Lemma (see [2, Theorem 1.12] for example) states that every subgroup of finite index in a finitely generated group is finitely generated. Hence M must be finitely generated. Furthermore, a characteristic subgroup of M is normal in G, so M must be characteristically simple. □

Theorem 2.3. *If $G \leq \mathrm{Sym}\,(\Omega)$ is primitive and infinite, with a finite point stabilizer G_α, then G has a unique minimal normal subgroup M, the centraliser $C_G(M)$ is trivial and M is a direct product*

$$M = \prod(K^g : g \in T),$$

where $K \leq M$ is some infinite non-abelian finitely generated simple group and $T \subseteq G_\alpha$ is a right transversal of the normaliser $N_G(K)$ in G.

Proof. If G is primitive and has a finite point-stabilizer, then G has only finitely many normal subgroups by Lemma 2.1 and Hall's Theorem ([5, Theorem 4]), and therefore satisfies min-n, the minimal condition on normal subgroups. By Theorem 2.2, M is characteristically simple and of finite index in G, and so by [17, Theorem A] M also satisfies min-n. The set $\{K \lhd M : K \neq \langle 1 \rangle\}$ therefore contains a minimal element, so there exists a minimal (non-trivial) normal subgroup K of M.

Let T be a right transversal of $N_G(K)$ in G. Since $G = MG_\alpha$, we may choose $T \subseteq G_\alpha$ with $1 \in T$; thus, the number of conjugates of K under elements of G is finite, and we may apply a well-known argument from the finite case ([3, Proof of Theorem 4.3A(iii)], for example) to deduce that M is a direct product of some of them, say $M = K_1 \times \cdots \times K_m$. Hence, K is infinite and simple, and is thus non-abelian. Since they are non-abelian and simple, the subgroups $K_1, \ldots, K_m$ are the only minimal normal subgroups of M ([3, Proof of Theorem 4.3A(iv)], for example), and therefore conjugation by elements of G permutes them amongst themselves. If $\{K_i : i \in I\}$ is any orbit of this action, then $\prod(K_i : i \in I)$ is a normal subgroup of G, and therefore must equal M. Thus G and G_α act transitively on $\{K_i : i \in I\}$, so $\{K_i : i \in I\} = \{K^g : g \in T\}$.

Finally we note that since M is finitely generated, K is finitely generated and because K is non-abelian, M is non-abelian and thus $C_G(M)$ is trivial. □

3. PROOF OF THEOREM1.1

We now examine separately the cases where M acts regularly on Ω, where M is non-regular and simple, and where M is non-regular and non-simple; these cases will correspond respectively to G being of type (i), (ii) or (iii) in Theorem 1.1. Since the descriptions of M are mutually exclusive, the same is true of the cases described in the theorem. Theorem 1.1 follows immediately from Theorem 2.3, and Theorems 3.1, 3.3 and 3.4, given below.

Theorem 3.1. *If $G \leq \mathrm{Sym}(\Omega)$ is an infinite primitive permutation group with a finite point stabilizer G_α and a unique minimal normal subgroup M that acts regularly on Ω, then $M = K_1 \times \cdots \times K_m$ where m is finite, $K_i \cong K$ for $1 \leq i \leq m$ for some infinite, non-abelian, finitely generated and simple group K, and G is equal to the split extension $M.G_\alpha$. Furthermore, no non-identity element of G_α induces an inner automorphism of M.*

Proof. By Theorem 2.3, G has a unique minimal normal subgroup M, and $M = K_1 \times \cdots \times K_m$, with $K_i \cong K$ for $1 \leq i \leq m$, for some infinite non-abelian finitely generated simple group K and some finite $m \geq 1$. Since $G_\alpha \cap M = M_\alpha = \langle 1 \rangle$ the extension $G = M.G_\alpha$ splits, and because $C_G(M) = \langle 1 \rangle$ no non-identity element of G_α induces an inner automorphism of M. □

In this case, we can identify Ω with M, and the natural action of G_α on Ω is permutation equivalent to the conjugation action of G_α on M. Primitive permutation groups with this structure have the following well-known characterisation.

Theorem 3.2. [3, Exercise 2.5.8] *If $G \leq \mathrm{Sym}(\Omega)$ and $N \lhd G$ acts regularly on Ω and $\alpha \in \Omega$, then G is primitive on Ω if and only if no non-trivial proper subgroup of N is normalised by G_α.* □

Peter Neumann has pointed out that under the conditions of Theorem 3.1 we can say a little more: there are no non-trivial proper $N_{G_\alpha}(K_1)$-invariant subgroups of K_1. Indeed, suppose $Y_1 \leq K_1$ is such a group. Let T be a right transversal $\{g_1, \ldots, g_m\}$ of $N_{G_\alpha}(K_1)$ in G_α, with elements of T labelled in such a way that $K_i = K_1^{g_i}$ for $1 \leq i \leq m$, and let $Y_i = Y_i^{g_i}$. Define $N := Y_1 \times \cdots \times Y_m$, a non-trivial proper subgroup of M. Since G_α permutes elements of the set $\{K_i : 1 \leq i \leq m\}$ transitively by conjugation, for each $g \in G_\alpha$ there exists a permutation $\sigma \in S_m$ such that for all $1 \leq i \leq m$ we have $g_i g = \overline{g} g_{\sigma(i)}$ for some $\overline{g} \in N_{G_\alpha}(K_1)$. Hence $Y_i^g = Y_1^{g_i g} = Y_1^{\overline{g} g_{\sigma(i)}} = Y_{\sigma(i)}$. Thus G_α normalises N and so G is not primitive by Theorem 3.2.

We now turn our attention to the case when M does not act regularly on Ω.

Theorem 3.3. *If $G \leq \mathrm{Sym}(\Omega)$ is an infinite primitive permutation group with a finite point stabilizer G_α and a unique minimal normal subgroup M that is non-regular and simple, then*

$$M \leq G \leq \mathrm{Aut}(M),$$

and M is a finitely generated, simple and non-abelian group of finite index in G. In particular, G is almost simple.

Proof. Suppose that M is simple and non-regular. By Theorem 2.3, M must be infinite, non-abelian, finitely generated and simple. Now G acts on M by conjugation, and by Theorem 2.3, $C_G(M)$ is trivial so this action is faithful. Thus G acts as a group of automorphisms, and we have $M \leq G \leq \mathrm{Aut}(M)$. □

Theorem 3.4. *If $G \leq \mathrm{Sym}\,(\Omega)$ is an infinite primitive permutation group with a finite point stabilizer and a non-regular, non-simple unique minimal normal subgroup, then G is permutation isomorphic to a subgroup of $H \,\mathrm{Wr}_\Delta\, \mathrm{Sym}\,(\Delta)$, acting in the product action on Γ^m, where $\Delta = \{1, \ldots, m\}$ and $H \leq \mathrm{Sym}\,(\Gamma)$ is an infinite primitive group with a finite point stabilizer and a non-regular simple unique minimal normal subgroup.*

Let $G \leq \mathrm{Sym}\,(\Omega)$ be an infinite primitive permutation group with a finite point stabilizer G_α and a non-simple minimal normal subgroup M. Since G_α is finite, it is closed in the usual permutation topology. It is easily seen that G and all subgroups of G are closed (see [8, pp. 519]). In particular M is a closed minimal normal subgroup of G. Furthermore, by Theorem 2.3, M also has a minimal normal subgroup K, and as previously noted K must be closed.

Thus we may reuse some of the arguments made in [8]. Our intention is to use parts of the proof of [8, Lemma 4.7]. Before doing so, we must first introduce some notation from [8, pp. 527–528], and state clearly the assumptions that have been made prior to the statement of [8, Lemma 4.7].

Fix $\alpha \in \Omega$. The *topological socle* of G (the group generated by the minimal closed normal subgroups of G) is denoted by B; in our case, B = M. Let N denote the normaliser $N_G(K)$, and for any $H \leq N$ let H^* be the subgroup of $\mathrm{Aut}\,(K)$ induced by H. Note that $N^* = K^* N_\alpha^*$. In [8] the set T agrees with our definition, given in Theorem 2.3, namely that $T \subseteq G_\alpha$ is a right transversal of $N_G(K)$ in G with $1 \in T$. If $(K_i : i \in I)$ is a family of groups, then the restricted direct product $\mathrm{res} \prod (K_i : i \in I)$ consists of sequences, indexed by I, of finite support. In [8, pp. 527], L is the group $\mathrm{res} \prod (K^g : g \in T)$, but in our case, because T is finite, L is the full direct product $\prod (K^g : g \in T)$, which is equal to M by Theorem 2.3. For $g \in T$ the function π_g is the projection from L to K^g, and R_g is defined to be $\pi_g(L_\alpha)$, with $R := R_1$. Immediately following this definition in [8, pp. 527], it is noted that when M is equal to the socle of G, the group R is trivial if and only if L_α is trivial.

Prior to the statement of [8, Lemma 4.7], several assumptions are made in the text. The topological assumptions can be safely ignored, as in our case G and all subgroups of G are closed. The other assumptions are that K is non-abelian and $L_\alpha \neq \langle 1 \rangle$. In our case K is always non-abelian by Theorem 2.3, and we have explicitly assumed that $(M; \Omega)$ is non-regular, thus guaranteeing $L_\alpha = M_\alpha \neq \langle 1 \rangle$. The statement of [8, Lemma 4.7] makes two further assumptions: that M is the topological socle of G, and that there exists $g \in T$ such that R_g is a proper subgroup of K^g. In our case the former is true because M is the unique minimal normal subgroup of G, and the latter is true because R_g is finite and K^g is infinite, for all $g \in T$. Note that R is non-trivial.

Proof of Theorem 3.4. Let Γ be the set of right cosets of $N_\alpha.C_G(K)$ in N and let γ be the element $N_\alpha.C_G(K) \in \Gamma$. By [8, Lemma 4.7, proof of claim (4)] we have that $K^* \cong K$ and $H := N^*$ acts primitively on Γ, with $H_\gamma \cong N_\alpha/(N_\alpha \cap C_G(K))$. Because N_α is finite, H_γ is finite. Since $H = N^* = K^* N_\alpha^*$ (as noted above), H is infinite. Thus, $(H; \Gamma)$ is an infinite primitive permutation group with finite point stabilizers.

We identify the subgroup K^* of H with K, so $K \leq H$, and define Z to be the group $\langle \mathrm{res} \prod (K^g : g \in T), H_\gamma \,\mathrm{Wr}\, \mathrm{Sym}\,(T) \rangle$, which in our case is equal to $H \,\mathrm{Wr}\, \mathrm{Sym}\,(T)$ because T is finite. Thus Z has a natural product action on Γ^T, the set of all functions from the finite set T to Γ. By [8, Lemma 4.7, claim (5)] (which contains a misprint: X should read Z) we have that $(G; \Omega)$ is permutation isomorphic to a subgroup of $(H \,\mathrm{Wr}\, \mathrm{Sym}\,(T); \Gamma^T)$.

It remains to check that H has a unique minimal normal subgroup that is simple and acts non-regularly on Γ. Since $K \leq H$, the simple group K is a minimal normal subgroup of H; by Theorem 2.2, it is the only one. Furthermore, in the proof of [8, Lemma 4.7, claim (4)] it is observed that $K_\gamma = R$. Since R is non-trivial, K is non-regular. □

This concludes our proof of Theorem 1.1. We remark briefly why types of groups present in the classification in [8] do not occur in Theorem 1.1. Groups of affine type (Type 1 in [8]) require K to be finite. Groups with a simple diagonal action (Type 4(a) in [8]) require K to be isomorphic to a point stabilizer in $\mathrm{soc}(G)$, impossible in our case because any stabilizer is finite. Groups satisfying $B = M$ with a product action in which H is diagonal (Type 4(b)(ii) in [8]) appear when the possibility that $R^g = K^g$ for all $g \in T$ is explored ([8, pp. 531]), but this cannot occur in our case because K is infinite and R is finite. Finally we repeat the remark in [8, pp. 534], that for finite primitive groups the twisted wreath product case in [6] occurs when the possibility that M is the unique minimal normal subgroup of G, and $(M; \Omega)$ is regular, and $K^* \leq N^*_\alpha$ is considered. Under these assumptions, [8, Lemma 4.12] implies that T must be infinite. In our case T is always finite, so twisted wreath products do not appear in our classification.

4. Examples

Given a prime $p > 10^{75}$, there exists a group T_p, often called a Tarski–Ol'Shanskiĭ Monster, such that every proper non-trivial subgroup of T_p has order p ([12, Theorem 28.1]). Any group T_p can be considered to be an infinite primitive permutation group with finite point stabilizers of type (ii). Indeed, let H be a proper non-trivial subgroup of T_p, and let Γ be the set $(T_p : H)$ of right cosets of H in T_p. Then T_p acts transitively and faithfully, but not regularly, on Γ by right multiplication. Any point stabilizer is isomorphic to H, a finite and maximal subgroup of T_p.

Constructing an example of type (iii) is trivial: let $(T_p; \Gamma)$ be as above, and let $\Delta = \{1, \ldots, n\}$ for some $n \geq 2$. The wreath product $T_p \,\mathrm{Wr}_\Gamma\, \mathrm{Sym}\,(\Delta)$ acting on Γ^n via the product action is primitive, and the point stabilizers are all finite.

Mark Sapir kindly pointed out ([14]) the existence of the following groups. For any prime $p > 10^{75}$, there exists a Tarski–Ol'Shanskiĭ Monster T_p such that T_p has a non-inner automorphism a and for all non-trivial $g \in T_p$ we have $\langle g, g^a \rangle = T_p$ and $g^{a^2} = g$. Note that this is not a property shared by every Tarski–Ol'Shanskiĭ Monster: Mark modified the construction of a Tarski–Ol'Shanskiĭ Monster so as to produce a Tarski–Ol'Shanskiĭ Monster group that possesses this non-inner automorphism.

We sketch the construction here. A Tarski–Ol'Shanskiĭ Monster is constructed by taking a quotient of the free group on two generators F_2. Let α be the obvious automorphism of order 2 interchanging the two generators of F_2. When constructing the Tarski–Ol'Shanskiĭ Monster quotient (see [12, pp 296–298]), each time a relation r is defined, we impose the additional relation $\alpha(r) = 1$. This yields a series of subgroups H_i of F_2, where H_i is generated by H_{i-1} and all these relations. Following the construction in [12], it is easy two see that any two elements in the resulting Tarski–Ol'Shanskiĭ Monster quotient will either generate the whole group or will commute and hence generate a cyclic group of finite order; so, to prove this construction does indeed yield a Tarski–Ol'Shanskiĭ Monster, we need only show that the resulting group is infinite. This is achieved by showing that at every stage we have a non-elementary hyperbolic group.

To see this, we first fix an enumeration $F_2 = \{g_1, g_2, \ldots\}$ of the elements of F_2. Now F_2 is non-elementary, torsion free and hyperbolic, and each H_i is finitely

generated and non-elementary. Thus (by [11, proof of Corollary 4]) every H_i is a *G-subgroup* or *Gromov subgroup* and the *quasiidentical relation* holds (see [11] for the definitions). Thus, (by [11, proof of Corollary 3]) there exists a sequence of epimorphisms $G = G_0 \to G_1 \to G_2 \cdots$ such that every G_i is a non-elementary hyperbolic group satisfying the quasiidentical relation, whose non-elementary subgroups are G-subgroups. Each element $g_i \in F_2$ has finite order in the group G_{2i-1} and the natural image of H_i in G_{2i} is an elementary subgroup of G_{2i} or coincides with G_{2i}.

Let G' be the limit of the G_i. The argument in [11, proof of Corollary 3] shows that G' is infinite and quasifinite. Our choice of relations force any non-trivial proper subgroup of G' to be cyclic of order p, so G' is a Tarski–Ol'Shanskiĭ Monster with a non-inner automorphism α of order 2.

These groups can be used to construct infinite primitive permutation groups with finite point stabilizers of type (i). Indeed, suppose T_p and a are as above, and let $A_2 = \langle a \rangle$. We define G to be the semidirect product $T_p \rtimes A_2$, where A_2 acts on T_p in the obvious way, and write this as a split extension $G = K.A$, where $T_p \cong K$ and $A_2 \cong A$. Now G acts faithfully and transitively (by right multiplication) on the set of right cosets $\Gamma = (G : A)$ of A in G, and if $\gamma \in \Gamma$ corresponds to the coset $A.1$, then $G_\gamma = A$. Since $K \lhd G$ we have $G = AK$, so K acts transitively on Γ, and $K_\gamma = \langle 1 \rangle$; hence K acts regularly on Γ. By construction, no subgroup of T_p is normalised by a, so Theorem 3.2 implies that $(G; \Gamma)$ is primitive.

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